Some subclasses of meromorphic multivalent functions involving a family of multiplier transforms

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Abstract. Making use of the principle of subordination between analytic functions and a family of multiplier transforms defined on the space of meromorphic functions, we introduce and investigate some new subclasses of meromorphic multivalent functions. Such results as inclusion relationships and integral-preserving properties associated with these subclasses are proved. Several subordination and superordination results involving this family of multiplier transforms are also investigated.

Keywords: analytic functions, meromorphic multivalent functions, subordination and superordination between analytic functions, Hadamard product (or convolution), multiplier transforms.

1. Introduction and preliminaries

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : \ z \in \mathbb{C} \text{ and } 0 < |z| < 1\} =: \mathbb{U} \backslash \{0\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number n and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a, n] := \{f \in \mathcal{H}(\mathbb{U}) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

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Let $f,\ g \in \Sigma_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^k =: (g * f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p\,(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition $\Re(p\,(z)) > 0 \quad (z \in \mathbb{U})$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write $f(z) \prec g(z), (z \in \mathbb{U})$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g\,(\omega(z))\,, (z \in \mathbb{U})$. Indeed, it is known that $f(z) \prec g(z), (z \in \mathbb{U}) \Longrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In a recent paper, El-Ashwah [6] defined the multiplier transform $\mathcal{D}_{\lambda,p}^{n,\,l}$ of functions $f \in \Sigma_p$ by

$$\mathcal{D}_{\lambda,p}^{n,\,l} f(z) := z^{-p} + \sum_{k=0}^{\infty} \left(\frac{\lambda + l(k+p)}{\lambda} \right)^n a_k z^k \tag{1.2}$$

$$(z \in \mathbb{U}^*;\ \lambda > 0;\ l \geqq 0;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ p \in \mathbb{N}).$$

It should be remarked that the operators $\mathcal{D}_{\lambda,1}^{n,1}$ and $\mathcal{D}_{1,1}^{n,1}$ are the multiplier transforms introduced and investigated, respectively, by Sarangi and Uralegaddi [14], and Uralegaddi and Somanatha [18, 19]. Analogous to $\mathcal{D}_{\lambda,p}^{n,\,l}$, we here define a new multiplier transform $\mathcal{I}_{\lambda,p,\mu}^{n,\,l}$ as follows.

By setting

$$f_{\lambda,p}^{n,\,l}(z) := z^{-p} + \sum_{k=0}^{\infty} \left(\frac{\lambda + l(k+p)}{\lambda} \right)^n z^k,$$

$$(z \in \mathbb{U}^*;\ n,\, l \geqq 0;\ \lambda > 0;\ p \in \mathbb{N}), \tag{1.3}$$

we define a new function $f_{\lambda,p,\mu}^{n,\,l}(z)$ in terms of the Hadamard product (or convolution):

$$f_{\lambda,p}^{n,l}(z) * f_{\lambda,p,\mu}^{n,l}(z) = \frac{1}{z(1-z)^{\mu}} \quad (z \in \mathbb{U}^*;\ \lambda,\ \mu > 0;\ n,\ l \geqq 0;\ p \in \mathbb{N}). \tag{1.4}$$

Then, analogous to $\mathcal{D}_{\lambda,p}^{n,l}$, we have

$$\mathcal{I}_{\lambda,p,\mu}^{n,l} f(z) := f_{\lambda,p,\mu}^{n,l}(z) * f(z), \quad (z \in \mathbb{U}^*;\ f \in \Sigma_p), \tag{1.5}$$

where (and throughout this paper unless otherwise mentioned) the parameters $n,\ l,\ p,\ \lambda$ and μ are constrained as follows:

$$n \geqq 0;\ l \geqq 0;\ p \in \mathbb{N},\ \lambda > 0 \text{ and } \mu > 0.$$

We can easily find from (1.3), (1.4) and (1.5) that

$$\mathcal{I}_{\lambda,p,\mu}^{n,l} f(z) = z^{-p} + \sum_{k=0}^{\infty} \frac{(\mu)_{k+1}}{(k+1)!} \left(\frac{\lambda}{\lambda + l(k+p)} \right)^n a_k z^k, \quad (z \in \mathbb{U}^*), \tag{1.6}$$

where $(\mu)_k$ is the Pochhammer symbol defined by

$$(\mu)_k := \begin{cases} 1, & (k=0), \\ \mu(\mu+1)\cdots(\mu+k-1), & (k \in \mathbb{N}). \end{cases}$$

Clearly, the operator $\mathcal{I}_{\lambda,1,\mu}^{n,1}$ $(n \in \mathbb{N}_0)$ is the well-known Cho-Kwon-Srivastava operator (see, for more details, [2, 3, 8, 13, 15]).

It is readily verified from (1.6) that

$$lz \left(\mathcal{I}_{\lambda,p,\mu}^{n+1,l} f \right)'(z) = \lambda \mathcal{I}_{\lambda,p,\mu}^{n,l} f(z) - (\lambda + p\,l) \mathcal{I}_{\lambda,p,\mu}^{n+1,l} f(z), \tag{1.7}$$

and

$$z \left(\mathcal{I}_{\lambda,p,\mu}^{n,l} f \right)'(z) = \mu \mathcal{I}_{\lambda,p,\mu+1}^{n,l} f(z) - (\mu+1) \mathcal{I}_{\lambda,p,\mu}^{n,l} f(z). \tag{1.8}$$

By making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{MS}_p^*(\eta;\phi)$, $\mathcal{MK}_p(\eta;\phi)$, $\mathcal{MC}_p(\eta,\delta;\phi,\psi)$ and $\mathcal{MQC}_p(\eta,\delta;\phi,\psi)$ of the class Σ_p which are defined by

$$\mathcal{MS}_p^*(\eta;\phi) := \left\{ f \in \Sigma_p :\ \frac{1}{p-\eta} \left(-\frac{zf'(z)}{f(z)} - \eta \right) \prec \phi(z), \right.$$
$$\left. (\phi \in \mathcal{P};\ 0 \leqq \eta < p\,;\ z \in \mathbb{U}) \right\},$$

$$\mathcal{MK}_p(\eta;\phi) := \left\{ f \in \Sigma_p :\ \frac{1}{p-\eta} \left(-1 - \frac{zf''(z)}{f'(z)} - \eta \right) \prec \phi(z), \right.$$
$$\left. (\phi \in \mathcal{P};\ 0 \leqq \eta < p\,;\ z \in \mathbb{U}) \right\},$$

$$\mathcal{MC}_p(\eta,\delta;\phi,\psi) := \left\{ f \in \Sigma_p :\ \exists\ g \in \mathcal{MS}_p^*(\eta;\phi) \right.$$
$$\text{such that } \frac{1}{p-\delta} \left(-\frac{zf'(z)}{g(z)} - \delta \right) \prec \psi(z)$$
$$\left. (\phi,\ \psi \in \mathcal{P};\ 0 \leqq \eta,\ \delta < p\,;\ z \in \mathbb{U}) \right\},$$

and

$$\mathcal{MQC}_p(\eta,\delta;\phi,\psi) := \Bigg\{ f \in \Sigma_p : \ \exists\, g \in \mathcal{MK}_p(\eta;\phi)$$
$$\text{such that } \ \frac{1}{p-\delta}\left(-\frac{(zf'(z))'}{g'(z)} - \delta\right) \prec \psi(z), (\phi,\ \psi \in \mathcal{P};\ 0 \leqq \eta,\ \delta < p;\ z \in \mathbb{U}) \Bigg\}.$$

Indeed, the above mentioned function classes are generalizations of the general meromorphic starlike, meromorphic convex, meromorphic close-to-convex and meromorphic quasi-convex functions in analytic function theory (see, for details, [1, 7, 11, 12, 16, 17, 20, 21, 22]).

Next, by using the operator defined by (1.6), we define the following subclasses $\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$, $\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$, $\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$ and $\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$ of the class Σ_p:

$$\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi) := \left\{ f \in \Sigma_p :\ \mathcal{I}^{n,l}_{\lambda,p,\mu} f \in \mathcal{MS}^*_p(\eta;\phi) \right\},$$
$$\mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi) := \left\{ f \in \Sigma_p :\ \mathcal{I}^{n,l}_{\lambda,p,\mu} f \in \mathcal{MK}_p(\eta;\phi) \right\},$$
$$\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) := \left\{ f \in \Sigma_p :\ \mathcal{I}^{n,l}_{\lambda,p,\mu} f \in \mathcal{MC}_p(\eta,\delta;\phi,\psi) \right\},$$

and

$$\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) := \left\{ f \in \Sigma_p :\ \mathcal{I}^{n,l}_{\lambda,p,\mu} f \in \mathcal{MQC}_p(\eta,\delta;\phi,\psi) \right\}.$$

Clearly, we know that

$$f \in \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi) \Longleftrightarrow -zf' \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi), \tag{1.9}$$

and

$$f \in \mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) \Longleftrightarrow -zf' \in \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi). \tag{1.10}$$

In order to prove our main results, we need the following definition and lemmas.

Definition 1. (See [10]) Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \left\{ \varepsilon \in \partial\mathbb{U} :\ \lim_{z\to\varepsilon} f(z) = \infty \right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$.

Lemma 1 ([5])**.** *Let $\kappa, \vartheta \in \mathbb{C}$. Suppose also that $\mathfrak{m}$ is convex and univalent in $\mathbb{U}$ with*

$$\mathfrak{m}(0) = 1, \quad \textit{and} \quad \Re(\kappa\mathfrak{m}(z) + \vartheta) > 0, \quad (z \in \mathbb{U}).$$

If $\mathfrak{u}$ *is analytic in* $\mathbb{U}$ *with* $\mathfrak{u}(0) = 1$*, then the subordination*

$$\mathfrak{u}(z) + \frac{z\mathfrak{u}'(z)}{\kappa\mathfrak{u}(z) + \vartheta} \prec \mathfrak{m}(z), \quad (z \in \mathbb{U})$$

implies that

$$\mathfrak{u}(z) \prec \mathfrak{m}(z), \quad (z \in \mathbb{U}).$$

Lemma 2 ([9])**.** *Let* h *be convex univalent in* $\mathbb{U}$ *and* ζ *be analytic in* $\mathbb{U}$ *with*

$$\Re(\zeta(z)) \geqq 0, \quad (z \in \mathbb{U}).$$

If q *is analytic in* $\mathbb{U}$ *and* $q(0) = h(0)$*, then the subordination*

$$q(z) + \zeta(z)zq'(z) \prec h(z), \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec h(z), \quad (z \in \mathbb{U}).$$

The main purpose of the present paper is to investigate some inclusion relationships and integral-preserving properties of the subclasses

$$\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi),\ \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi),\ \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) \text{ and } \mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$$

of meromorphic functions involving the operator $\mathcal{I}^{n,l}_{\lambda,p,\mu}$. Several subordination and superordination results involving this operator are also investigated.

2. The main inclusion relationships

We begin by presenting our first inclusion relationship given by Theorem 1 below.

Theorem 1. *Let* $0 \leqq \eta < p$ *and* $\phi \in \mathcal{P}$ *with*

$$\max_{z\in\mathbb{U}} \{\Re(\phi(z))\} < \min\left\{\frac{\mu+1-\eta}{p-\eta},\ \frac{\lambda+pl-\eta l}{(p-\eta)l}\right\}, \quad (z \in \mathbb{U}). \tag{2.1}$$

Then

$$\mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi) \subset \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi) \subset \mathcal{MS}^{n+1,l}_{\lambda,p,\mu}(\eta;\phi).$$

Proof. We first prove that

$$\mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi) \subset \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi). \tag{2.2}$$

Let $f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi)$ and suppose that

$$\mathfrak{h}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)} - \eta\right), \tag{2.3}$$

where $\mathfrak{h}$ is analytic in $\mathbb{U}$ with $\mathfrak{h}(0) = 1$. Combining (1.8) and (2.3), we find that

$$\mu \frac{\mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f(z)}{\mathcal{I}^{n,\,l}_{\lambda,p,\mu} f(z)} = -(p-\eta)\mathfrak{h}(z) - \eta + \mu + 1. \tag{2.4}$$

Taking the logarithmical differentiation on both sides of (2.4) and multiplying the resulting equation by z, we get

$$\begin{aligned} &\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f\right)'(z)}{\mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f(z)} - \eta\right) \\ &= \mathfrak{h}(z) + \frac{z\mathfrak{h}'(z)}{-(p-\eta)\mathfrak{h}(z) - \eta + \mu + 1} \prec \phi(z). \end{aligned} \tag{2.5}$$

By virtue of (2.1), an application of Lemma 1 to (2.5) yields $\mathfrak{h} \prec \phi$, that is $f \in \mathcal{MS}^{n,\,l}_{\lambda,p,\mu}(\eta;\phi)$. Thus, the assertion (2.2) of Theorem 1 holds.

To prove the second part of Theorem 1, we assume that $f \in \mathcal{MS}^{n,\,l}_{\lambda,p,\mu}(\eta;\phi)$ and set

$$\mathfrak{g}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n+1,\,l}_{\lambda,p,\mu} f\right)'(z)}{\mathcal{I}^{n+1,\,l}_{\lambda,p,\mu} f(z)} - \eta\right), \tag{2.6}$$

where $\mathfrak{g}$ is analytic in $\mathbb{U}$ with $\mathfrak{g}(0) = 1$. Combining (1.7), (2.1) and (2.6) and applying the similar method of proof of the first part, we get $\mathfrak{g} \prec \phi$, that is $f \in \mathcal{MS}^{n+1,\,l}_{\lambda,p,\mu}(\eta;\phi)$. Therefore, the second part of Theorem 1 also holds. The proof of Theorem 1 is evidently completed. □

Theorem 2. *Let* $0 \leqq \eta < p$ *and* $\phi \in \mathcal{P}$ *with* (2.1) *holds. Then*

$$\mathcal{MK}^{n,\,l}_{\lambda,p,\mu+1}(\eta;\phi) \subset \mathcal{MK}^{n,\,l}_{\lambda,p,\mu}(\eta;\phi) \subset \mathcal{MK}^{n+1,\,l}_{\lambda,p,\mu}(\eta;\phi).$$

Proof. In view of (1.9) and Theorem 1, we find that

$$\begin{aligned} f \in \mathcal{MK}^{n,\,l}_{\lambda,p,\mu+1}(\eta;\phi) &\Longleftrightarrow \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f \in \mathcal{MK}_p(\eta;\phi) \\ &\Longleftrightarrow -z\left(\mathcal{I}^{n,\,l}_{\lambda,p,\mu+1} f\right)' \in \mathcal{MS}^*_p(\eta;\phi) \\ &\Longleftrightarrow \mathcal{I}^{n,\,l}_{\lambda,p,\mu+1}\left(-zf'\right) \in \mathcal{MS}^*_p(\eta;\phi) \\ &\Longleftrightarrow -zf' \in \mathcal{MS}^{n,\,l}_{\lambda,p,\mu+1}(\eta;\phi) \\ &\Longrightarrow -zf' \in \mathcal{MS}^{n,\,l}_{\lambda,p,\mu}(\eta;\phi) \\ &\Longleftrightarrow \mathcal{I}^{n,\,l}_{\lambda,p,\mu}\left(-zf'\right) \in \mathcal{MS}^*_p(\eta;\phi) \\ &\Longleftrightarrow -z\left(\mathcal{I}^{n,\,l}_{\lambda,p,\mu} f\right)' \in \mathcal{MS}^*_p(\eta;\phi) \\ &\Longleftrightarrow \mathcal{I}^{n,\,l}_{\lambda,p,\mu} f \in \mathcal{MK}_p(\eta;\phi) \\ &\Longleftrightarrow f \in \mathcal{MK}^{n,\,l}_{\lambda,p,\mu}(\eta;\phi), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} f \in \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta;\phi) &\Longleftrightarrow -zf' \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi) \\ &\Longrightarrow -zf' \in \mathcal{MS}^{n+1,l}_{\lambda,p,\mu}(\eta;\phi) \\ &\Longleftrightarrow \mathcal{I}^{n+1,l}_{\lambda,p,\mu}\left(-zf'\right) \in \mathcal{MS}^{*}_{p}(\eta;\phi) \\ &\Longleftrightarrow \mathcal{I}^{n+1,l}_{\lambda,p,\mu} f \in \mathcal{MK}_{p}(\eta;\phi) \\ &\Longleftrightarrow f \in \mathcal{MK}^{n+1,l}_{\lambda,p,\mu}(\eta;\phi). \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we deduce that the assertion of Theorem 2 holds. □

Theorem 3. *Let* $0 \leqq \eta < p$, $0 \leqq \delta < p$ *and* $\phi,\ \psi \in \mathcal{P}$ *with* (2.1) *holds. Then*

$$\mathcal{MC}^{n,l}_{\lambda,p,\mu+1}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}^{n+1,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi).$$

Proof. We begin by proving that

$$\mathcal{MC}^{n,l}_{\lambda,p,\mu+1}(\eta,\delta;\phi,\psi) \subset \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi). \tag{2.9}$$

Let $f \in \mathcal{MC}^{n,l}_{\lambda,p,\mu+1}(\eta,\delta;\phi,\psi)$. Then, by definition, we know that

$$\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z)}-\delta\right) \prec \psi(z), \quad (z \in \mathbb{U}) \tag{2.10}$$

with $g \in \mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta;\phi)$.

Moreover, by Theorem 1, we know that $g \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$, which implies that

$$\mathrm{q}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}g\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)}-\eta\right) \prec \phi(z), \quad (z \in \mathbb{U})\,. \tag{2.11}$$

We now suppose that

$$\mathrm{p}(z) := \frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)}-\delta\right), \quad (z \in \mathbb{U})\,, \tag{2.12}$$

where p is analytic in $\mathbb{U}$ with $\mathrm{p}(0) = 1$. Combining (1.8) and (2.12), we find that

$$-[(p-\delta)\mathrm{p}(z)+\delta]\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z) = \mu\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f(z) - (\mu+1)\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z). \tag{2.13}$$

Differentiating both sides of (2.13) with respect to z and multiplying the resulting equation by z, we get

$$\begin{aligned} &-(p-\delta)z\mathbb{p}'(z)-[(p-\delta)\mathbb{p}(z)+\delta][-(p-\eta)\mathbb{q}(z)-\eta+\mu+1] \\ &=\mu\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)}. \end{aligned} \tag{2.14}$$

In view of (1.8), (2.11) and (2.14), we conclude that

$$\begin{aligned} &\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z)}-\delta\right) \\ &=\mathbb{p}(z)+\frac{z\mathbb{p}'(z)}{-(p-\eta)\mathbb{q}(z)-\eta+\mu+1}\prec\psi(z), \quad (z\in\mathbb{U}). \end{aligned} \tag{2.15}$$

By noting that (2.1) holds and

$$\mathbb{q}(z)\prec\phi(z), \quad (z\in\mathbb{U}),$$

we know that

$$\Re(-(p-\eta)\mathbb{q}(z)-\eta+\mu+1)>0, \quad (z\in\mathbb{U}).$$

Thus, an application of Lemma 2 to (2.15) yields

$$\mathbb{p}(z)\prec\psi(z), \quad (z\in\mathbb{U}),$$

that is, that $f\in\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$, which implies that the assertion (2.9) of Theorem 3 holds.

By virtue of (1.7) and (2.1), and making use of the similar arguments of the details above, we deduce that

$$\mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)\subset\mathcal{MC}^{n+1,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi).$$

The proof of Theorem 3 is thus completed. □

Theorem 4. *Let* $0\leqq\eta<p$, $0\leqq\delta<p$ *and* $\phi,\ \psi\in\mathcal{P}$ *with* (2.1) *holds. Then*

$$\mathcal{MQC}^{n,l}_{\lambda,p,\mu+1}(\eta,\delta;\phi,\psi)\subset\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)\subset\mathcal{MQC}^{n+1,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi).$$

Proof. In view of (1.10) and Theorem 3, and by similarly applying the method of proof of Theorem 2, we conclude that the assertion of Theorem 4 holds. □

3. A set of integral-preserving properties

In this section, we derive some integral-preserving properties involving two families of integral operators.

Theorem 5. *Let $f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$ with $\phi \in \mathcal{P}$ and*

$$\Re(\phi(z)) < \frac{\Re(\nu)-\eta}{p-\eta}, \quad (z \in \mathbb{U};\ \Re(\nu) > p). \tag{3.1}$$

Then the integral operator $F_\nu(f)$ defined by

$$F_\nu(f) := F_\nu(f)(z) = \frac{\nu-p}{z^\nu}\int_0^z t^{\nu-1} f(t)dt, \quad (z \in \mathbb{U};\ \Re(\nu) > p) \tag{3.2}$$

belongs to the class $\mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$.

Proof. Let $f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta;\phi)$. Then, from (3.2), we find that

$$z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)\right)'(z) + \nu\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z) = (\nu-p)\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z). \tag{3.3}$$

By setting

$$\mathbb{P}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z)} - \eta\right), \tag{3.4}$$

we observe that $\mathbb{P}$ is analytic in $\mathbb{U}$ with $\mathbb{P}(0) = 1$. It follows from (3.3) and (3.4) that

$$-(p-\eta)\mathbb{P}(z) - \eta + \nu = (\nu-p)\frac{\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z)}. \tag{3.5}$$

Differentiating both sides of (3.5) with respect to z logarithmically and multiplying the resulting equation by z, we get

$$\begin{aligned}&\mathbb{P}(z) + \frac{z\mathbb{P}'(z)}{-(p-\eta)\mathbb{P}(z) - \eta + \nu}\\ &= \frac{1}{p-\eta}\left(-\frac{z(\mathcal{I}^{n,l}_{\lambda,p,\mu}f)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)} - \eta\right) \prec \phi(z), \quad (z \in \mathbb{U}).\end{aligned} \tag{3.6}$$

Since (3.1) holds, an application of Lemma 1 to (3.6) yields

$$\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}F_\nu(f)(z)} - \eta\right) \prec \phi(z),$$

which implies that the assertion of Theorem 5 holds. □

Theorem 6. *Let $f \in \mathcal{MK}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$ with $\phi \in \mathcal{P}$ and* (3.1) *holds. Then the integral operator $F_\nu(f)$ defined by* (3.2) *belongs to the class $\mathcal{MK}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$.*

Proof. By virtue of (1.9) and Theorem 5, we easily find that

$$\begin{aligned} f \in \mathcal{MK}_{\lambda,p,\mu}^{n,l}(\eta;\phi) &\Longleftrightarrow -zf' \in \mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi) \\ &\Longrightarrow F_\nu\left(-zf'\right) \in \mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi) \\ &\Longleftrightarrow -z\left(F_\nu(f)\right)' \in \mathcal{MS}_p^*(\eta;\phi) \\ &\Longleftrightarrow F_\nu(f) \in \mathcal{MK}_{\lambda,p,\mu}^{n,l}(\eta;\phi). \end{aligned}$$

The proof of Theorem 6 is evidently completed. □

Theorem 7. *Let $f \in \mathcal{MC}_{\lambda,\mu}^{n}(\eta,\delta;\phi,\psi)$ with $\phi \in \mathcal{P}$ and* (3.1) *holds. Then the integral operator $F_\nu(f)$ defined by* (3.2) *belongs to the class $\mathcal{MC}_{\lambda,\mu}^{n}(\eta,\delta;\phi,\psi)$.*

Proof. Let $f \in \mathcal{MC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi)$. Then, by definition, we know that there exists a function $g \in \mathcal{MS}_p^*(\eta;\phi)$ such that

$$\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l}f\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}g(z)}-\eta\right) \prec \psi(z), \quad (z \in \mathbb{U})\,. \tag{3.7}$$

Since $g \in \mathcal{MS}_p^*(\eta;\phi)$, by Theorem 5, we easily find that $F_\nu(g) \in \mathcal{MS}_p^*(\eta;\phi)$, which implies that

$$\mathbb{H}(z) := \frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l}F_\nu(g)\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}F_\nu(g)(z)}-\eta\right) \prec \phi(z). \tag{3.8}$$

We now set

$$\mathbb{Q}(z) := \frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l}F_\nu(f)\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}F_\nu(g)(z)}-\delta\right), \tag{3.9}$$

where $\mathbb{Q}$ is analytic in $\mathbb{U}$ with $\mathbb{Q}(0)=1$. From (3.3) and (3.9), we get

$$-[(p-\delta)\mathbb{Q}(z)+\delta]\mathcal{I}_{\lambda,p,\mu}^{n,l}F_\nu(g)(z)+\nu\mathcal{I}_{\lambda,p,\mu}^{n,l}F_\nu(f)(z) = (\nu-p)\mathcal{I}_{\lambda,p,\mu}^{n,l}f(z). \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we find that

$$\begin{aligned} &-(p-\delta)z\mathbb{Q}'(z)-[(p-\delta)\mathbb{Q}(z)+\delta][-(p-\eta)\mathbb{H}(z)-\eta+\nu] \\ &= (\nu-p)\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l}f\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}F_\nu(g)(z)}. \end{aligned} \tag{3.11}$$

By virtue of (1.8), (3.8) and (3.11), we deduce that

$$\begin{aligned}&\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l}f\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}g(z)}-\delta\right)\\ &=\mathbb{Q}(z)+\frac{z\mathbb{Q}'(z)}{-(p-\eta)\mathbb{H}(z)-\eta+\nu}\prec\psi(z),\quad (z\in\mathbb{U}).\end{aligned}\tag{3.12}$$

The remainder of the proof of Theorem 7 is much akin to that of Theorem 3. We, therefore, choose to omit the analogous details involved. We thus find that

$$\mathbb{Q}(z)\prec\psi(z),\quad (z\in\mathbb{U}),$$

which implies that $F_\nu(f)\in\mathcal{MC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi)$. The proof of Theorem 7 is thus completed. □

Theorem 8. *Let $f\in\mathcal{MQC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi)$ with $\phi\in\mathcal{P}$ and (3.1) holds. Then the integral operator $F_\nu(f)$ defined by (3.2) belongs to the class $\mathcal{MQC}_{\lambda,p,\mu}^{n,l}(\eta,\delta;\phi,\psi)$.*

Proof. In view of (1.10) and Theorem 7, and by similarly applying the method of proof of Theorem 6, we deduce that the assertion of Theorem 8 holds. □

Theorem 9. *Let $f\in\mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$ with $\phi\in\mathcal{P}$ and*

$$\Re\left(\sigma-\eta\,\xi-(p-\eta)\xi\,\phi(z)\right)>0,\quad (z\in\mathbb{U};\ \xi\neq 0).\tag{3.13}$$

Then the function $\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)\in\Sigma_p$ defined by

$$\begin{aligned}\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f):=&\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)(z)\\ &=\left(\frac{\sigma-p\,\xi}{z^\sigma}\int_0^z t^{\sigma-1}\left(\mathcal{I}_{\lambda,\mu}^{n}f(t)\right)^\xi dt\right)^{1/\xi},\quad (z\in\mathbb{U}^*;\ \xi\neq 0)\end{aligned}\tag{3.14}$$

belongs to the class $\mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$.

Proof. Let $f\in\mathcal{MS}_{\lambda,p,\mu}^{n,l}(\eta;\phi)$ and suppose that

$$\mathbb{M}(z):=\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)(z)}-\eta\right),\quad (z\in\mathbb{U}).\tag{3.15}$$

Combining (3.14) and (3.15), we have

$$\sigma-\eta\,\xi-(p-\eta)\xi\,\mathbb{M}(z)=(\sigma-p\,\xi)\left(\frac{\mathcal{I}_{\lambda,p,\mu}^{n,l}f(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,l}K_\xi^\sigma(f)(z)}\right)^\xi.\tag{3.16}$$

Making use of (3.14), (3.15) and (3.16), we get

$$\begin{aligned}
&\mathbb{M}(z)+\frac{z\mathbb{M}'(z)}{\sigma-\eta\,\xi-(p-\eta)\xi\,\mathbb{M}(z)}\\
&=\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,\,l}f\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,\,l}f(z)}-\eta\right)\prec\phi(z),\quad(z\in\mathbb{U}).
\end{aligned}\tag{3.17}$$

Since (3.13) holds, an application of Lemma 1 to (3.17) yields

$$\mathbb{M}(z)\prec\phi(z),\quad(z\in\mathbb{U}),$$

that is, that $\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(f)\in\mathcal{MS}_{\lambda,p,\mu}^{n,\,l}(\eta;\phi)$. We thus complete the proof of Theorem 9. □

Theorem 10. *Let $f\in\mathcal{MK}_{\lambda,p,\mu}^{n,\,l}(\eta;\phi)$ with $\phi\in\mathcal{P}$ and (3.13) holds. Then the function $\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(f)\in\Sigma_p$ defined by (3.14) belongs to the class $\mathcal{MK}_{\lambda,p,\mu}^{n,\,l}(\eta;\phi)$.*

Proof. By virtue of (1.9) and Theorem 9, and by similarly applying the method of proof of Theorem 6, we conclude that the assertion of Theorem 10 holds. □

Theorem 11. *Let $f\in\mathcal{MC}_{\lambda,p,\mu}^{n,\,l}(\eta,\delta;\phi,\psi)$ with $\phi\in\mathcal{P}$ and (3.13) holds.*

Then the function $\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(f)\in\Sigma_p$ defined by (3.14) belongs to the class $\mathcal{MC}_{\lambda,p,\mu}^{n,\,l}(\eta,\delta;\phi,\psi)$.

Proof. Let $f\in\mathcal{MC}_{\lambda,p,\mu}^{n,\,l}(\eta,\delta;\phi,\psi)$. Then, by definition, we know that there exists a function $g\in\mathcal{MS}_p^*(\eta;\phi)$ such that (3.7) holds. Since $g\in\mathcal{MS}_p^*(\eta;\phi)$, by Theorem 9, we easily find that $\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(g)\in\mathcal{MS}_p^*(\eta;\phi)$, which implies that

$$\mathbb{R}(z):=\frac{1}{p-\eta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(g)\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(g)(z)}-\eta\right)\prec\phi(z).\tag{3.18}$$

We now set

$$\mathbb{L}(z):=\frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(f)\right)'(z)}{\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(g)(z)}-\delta\right),\tag{3.19}$$

where $\mathbb{L}$ is analytic in $\mathbb{U}$ with $\mathbb{L}(0)=1$. From (3.14) and (3.19), we get

$$-\xi[(p-\delta)\mathbb{L}(z)+\delta]\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(g)(z)+\delta\mathcal{I}_{\lambda,p,\mu}^{n,\,l}K_\xi^\sigma(f)(z)=(\delta-p\,\xi)\mathcal{I}_{\lambda,p,\mu}^{n,\,l}f(z).\tag{3.20}$$

Combining (3.18), (3.19) and (3.20), we find that

$$\begin{aligned} & -\xi(p-\delta)z\mathbb{L}'(z) - [(p-\delta)\mathbb{L}(z)+\delta][-(p-\eta)\xi\mathbb{R}(z) - \eta\xi + \delta] \\ & = (\delta - p\xi)\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(g)(z)}. \end{aligned} \tag{3.21}$$

By virtue of (1.8), (3.18) and (3.21), we deduce that

$$\begin{aligned} & \frac{1}{p-\delta}\left(-\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)} - \delta\right) \\ & = \mathbb{L}(z) + \frac{z\mathbb{L}'(z)}{-(p-\eta)\xi\mathbb{R}(z) - \eta\xi + \delta} \prec \psi(z), \quad (z \in \mathbb{U}). \end{aligned} \tag{3.22}$$

The remainder of the proof of Theorem 11 is similar to that of Theorem 3. We, therefore, choose to omit the analogous details involved. We thus find that

$$\mathbb{L}(z) \prec \psi(z), \quad (z \in \mathbb{U}),$$

which implies that $\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f) \in \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$. The proof of Theorem 11 is thus completed. □

Theorem 12. *Let $f \in \mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$ with $\phi \in \mathcal{P}$ and (3.13) holds. Then the function $\mathcal{I}^{n,l}_{\lambda,p,\mu}K^{\sigma}_{\xi}(f) \in \Sigma_p$ defined by (3.14) belongs to the class $\mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta,\delta;\phi,\psi)$.*

Proof. By virtue of (1.10) and Theorem 11, and by similarly applying the method of proof of Theorem 6, we deduce that the assertion of Theorem 12 holds. □

4. Subordination and superordination results

Finally, we derive some subordination and superordination results associated with the operator $\mathcal{I}^{n,l}_{\lambda,p,\mu}$. The proofs are much akin to that of the results obtained by Cho *et al.* [4], we here choose to omit the details involved.

Corollary 1. *Let $f,\ g \in \Sigma_p$ and $l > 0$. If*

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho, \quad \left(z \in \mathbb{U};\ \varphi(z) := z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)\right), \tag{4.1}$$

where

$$\varrho := \frac{l^2+\lambda^2-|l^2-\lambda^2|}{4l\lambda}, \tag{4.2}$$

then the following subordination relationship

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g(z), \quad (z \in \mathbb{U}).$$

Furthermore, the function $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g$ *is the best dominant.*

Corollary 2. *Let* $f,\ g \in \Sigma_p$ *. If*

$$\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right) > -\varpi, \quad \left(z \in \mathbb{U};\ \chi(z) := z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z)\right),$$

where

$$\varpi := \frac{1+\mu^2-|1-\mu^2|}{4\mu}, \tag{4.3}$$

then the following subordination relationship

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z), \quad (z \in \mathbb{U}).$$

Furthermore, the function $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g$ *is the best dominant.*

If f is subordinate to $\mathscr{F}$, then $\mathscr{F}$ is superordinate to f. We now derive the following superordination results.

Corollary 3. *Let* $f,\ g \in \Sigma_p$ *and* $l > 0$*. If*

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho, \quad \left(z \in \mathbb{U};\ \varphi(z) := z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z)\right),$$

where ϱ *is given by* (4.2)*, also let the function* $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f$ *is univalent in* $\mathbb{U}$ *and* $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f \in Q$*, then the following subordination relationship*

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z), \quad (z \in \mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g(z) \prec z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f(z), \quad (z \in \mathbb{U}).$$

Furthermore, the function $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g$ *is the best subordinant.*

Corollary 4. *Let* $f,\ g \in \Sigma_p$. *If*

$$\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right) > -\varpi, \quad \left(z\in\mathbb{U};\ \chi(z) := z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z)\right),$$

where ϖ *is given by* (4.3), *also let the function* $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f$ *is univalent in* $\mathbb{U}$ *and* $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f \in Q$, *then the following subordination relationship*

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f(z), \quad (z\in\mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z), \quad (z\in\mathbb{U}).$$

Furthermore, the function $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g$ *is the best subordinant.*

Combining the above mentioned subordination and superordination results involving the operator $\mathcal{I}^{n,l}_{\lambda,p,\mu}$, we get the following "sandwich-type results".

Corollary 5. *Let* $f,\ g_k \in \Sigma_p$ $(k=1,\,2)$ *and* $l>0$. *If*

$$\Re\left(1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\varrho, \quad \left(z\in\mathbb{U};\ \varphi_k(z) := z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g_k(z) \quad (k=1,\,2)\right),$$

where ϱ *is given by* (4.2), *also let the function* $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f$ *is univalent in* $\mathbb{U}$ *and* $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f \in Q$, *then the subordination relationship*

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g_1(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g_2(z), \quad (z\in\mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g_1(z) \prec z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g_2(z), \quad (z\in\mathbb{U}).$$

Furthermore, the functions $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g_1$ *and* $z^p\mathcal{I}^{n+1,l}_{\lambda,p,\mu}g_2$ *are, respectively, the best subordinant and the best dominant.*

Corollary 6. *Let* $f,\ g_k \in \Sigma_p$ $(k=1,\,2)$. *If*

$$\Re\left(1+\frac{z\chi_k''(z)}{\chi_k'(z)}\right) > -\varpi, \quad \left(z\in\mathbb{U};\ \chi_k(z) := z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g_k(z) \quad (k=1,\,2)\right),$$

where ϖ *is given by* (4.3), *also let the function* $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f$ *is univalent in* $\mathbb{U}$ *and* $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f \in Q$, *then the subordination relationship*

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g_1(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu+1}g_2(z), \quad (z\in\mathbb{U})$$

implies that

$$z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g_1(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z) \prec z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g_2(z), \quad (z\in\mathbb{U}).$$

Furthermore, the functions $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g_1$ *and* $z^p\mathcal{I}^{n,l}_{\lambda,p,\mu}g_2$ *are, respectively, the best subordinant and the best dominant.*

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